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On Perpetuants.

By Captain P. A. MacMahon, R. A.

Professor Cayley has recently treated the subject of Perpetuants in the *American Journal of Mathematics*, and has largely developed the theory of the syzygies existing between them. I here make a few remarks upon the general subject, and then proceed to carry on the theory from the point where it was left by Professor Cayley.

The particular result that I obtain is the discovery that the simplest sextic perpetuant is of weight thirty-one; this result is, as will be seen, a somewhat remarkable one, since the *prima facie* probability was much against there being one of so low a weight. Postponing all explanations of the language made use of, the way in which this result comes out is as follows: for the weight $31 - 6$, $= 25$, the number of exemplar quintic perpetuants $\left(\text{coeff. of } x^{25} \text{ in } \frac{x^{15}}{\mathbf{2.3.4.5}}\right)$ is $= 7$, viz. these are $\overbrace{112},\ \overbrace{114},\ \overbrace{122},\ \overbrace{124},\ \overbrace{132},\ \overbrace{213},\ \overbrace{312},$
but it appears from the discussion of the syzygies of the third kind, that these are not each of them a sextic syzygant, but that only the combinations

$$\overbrace{112},\ \overbrace{114},\ \overbrace{122},\ \overbrace{132},\quad \overbrace{213} + 2.\overbrace{124},\ \overbrace{312} - 2.\overbrace{124},$$

are each of them a sextic syzygant, viz. the number of these is $= 6$. Hence for the weight 31 the number of sextic perpetuants is $7 - 6, = 1$.

Of a certain degree λ and weight w, there exist in general perpetuant forms of two kinds, which may be called *exemplar* and *non-exemplar;* certain linear relations subsist between them, so that the non-exemplar forms are reducible by aid of the exemplar forms; of the second, third and fourth degrees, every form is exemplar, the simplest forms being symbolized as is well known by the partitions 2, 3, 43, respectively; for the fifth degree, we have for the lowest weight fifteen, one exemplar form 543^2, two non-exemplar $5^2 32$, 5^3, the exemplar form being the simplest of the three; but there is another reason for choosing this form as the representative, for it will be observed that the symbol 543^2 contains in itself the symbol 43 of the simplest quartic perpetuant, and it will be proved that it is proper to take for the exemplar symbol of a perpetuant of any given degree, that one which contains the exemplar symbol of the degree next below, and that such forms are in fact the only ones that it is necessary to consider, the remainder being certainly non-exemplar.

For suppose the form $\lambda\mu\nu\pi\ldots$ of degree λ to be not a perpetuant, that is to be reducible by aid of compound forms and exemplar forms of lower degrees; then by the process, named by Prof. Cayley 'capitation,' it is at once obvious that $\varkappa \geqq \lambda$, $\varkappa\lambda\mu\nu\pi\ldots$ is reducible, at the worst, by aid of $\varkappa^{\text{ic}}$ perpetuants, involving lower exemplar symbols; for instance consider the form 532; we have

$$532 = 3^2 2.2 - 43^2 - 2(3^2 2^2),$$

whence $5^2 32 = 3^3 2.2^2 - 543^2 - 2(3^3 2^2.2) +$ terms of lower degree; this equation may be operated upon in the same manner any number of times by a capitation of any degree $\nless 5$, proving that any form thus obtained on the left-hand side is reducible by means of exemplar forms, simply because 532 is not a perpetuant.

It follows that every sextic exemplar form must contain in itself the symbol 543^2 and that it is unnecessary to consider other forms.

Another useful principle is, that if a form $\lambda\mu\nu\pi\ldots$ of degree λ is reducible, then, by the reverse process of decapitation, it follows that the form $\mu\nu\pi\ldots$ must be expressible as a sextic syzygant; note that we must have $\lambda = \mu + 1$, as otherwise $\lambda\mu\nu\pi\ldots$ would be certainly reducible.

Thus to every perpetuant of degree μ, that is not a $(\mu+1)^{\text{ic}}$ syzygant, will certainly correspond a $(\mu+1)^{\text{ic}}$ perpetuant, by simply capitating the μ^{ic} symbol with the part $\mu+1$; *ex. gr.* we know that the generating function for quintic syzygies is $\frac{x^7}{1-x^2.1-x^4}$, and that each syzygy involves a quartic perpetuant whose symbol contains one and only one part 3, the number of syzygies being exactly equal to the number of such quartic forms; consequently, no quartic form containing more than one part 3 can possibly be a quintic syzygant, and therefore 543^2 must be the simplest exemplar form of the fifth degree. The question is then in reality, not to determine the total generating function for the syzygies, but rather to find it for those syzygies which involve the exemplar perpetuant forms of the degree next below; thus, representing $1-x^\mu$ by μ for brevity, the generating function for μ^{ic} perpetuants is $\frac{x^\mu}{\mu}\{$G. F. for $(\mu-1)^{\text{ic}}$

perpetuants — G. F. for μ^{ic} syzygies which involve $(\mu-1)^{ic}$ perpetuants}; *ex. gr.* for degree 5, we arrive at the generating function

$$\frac{x^5}{5}\left(\frac{x^7}{2.3.4}-\frac{x^7}{2.4}\right)=\frac{x^{15}}{2.3.4.5},$$

the well-known result; and for degree 6, it will be $\frac{x^6}{6}\left(\frac{x^{15}}{2.3.4.5}-\omega(x)\right)$, wherein $\omega(x)$ is to be determined. There appears to be another way of regarding the question which it may be useful to mention.

It is a remarkable fact that there is an exact correspondence between the reducible quartic forms and the quintic syzygies which involve quartic perpetuants (which are in this case the whole of the syzygies); and what is still more strange is that presuming this same correspondence to hold between the reducible quintic forms and the sextic syzygies, a result is reached which exactly accords with that rigorously obtained in the sequel. This is not all, for the quintic syzygies can be derived from the reduced expressions of the non-perpetuant quartic forms by a direct operation. I have (Proc. L. M. S. Vol. XV, p. 31) explained the laws of a series of inverse operators, and it is by means of these that the derivation is effected; if $d_0, d_1, \dots d_\lambda$ be Mr. Hammond's operators, then the type of the operators referred to is

$$V_{-\kappa}=(\kappa)\,d_0-(\kappa+1)\,d_1+(\kappa+2)\,d_2-\dots(-)^\lambda(\kappa+\lambda)\,d_\lambda+\dots$$

$(\kappa+\lambda)$ being symbolic of a symmetric function; consider now the reduction of the quartic form (4), viz. $(4)=(2)^2-2\,(2^2)$,
and operating on both sides of this with V_{-3}, we get at once

$$(43)+(32^2)=32.2-3.2^2,$$

which is the simplest quintic syzygy; in like manner the operator $V_{-3^{\sigma}}$ gives the whole of the quintic syzygies, and each involves of necessity a quartic perpetuant; this follows of course from the occurrence of the number 3 in the operator V_{-3}.

It follows that G. F. for quintic perpetuants $=\frac{x^5}{5}\cdot x^3\cdot\frac{x^7}{2.3.4}=\frac{x^{15}}{2.3.4.5}$. Now, just as by superposing any quartic form with the symbolic number 3 (3 being the simplest cubic form which is not a quartic syzygant) we obtain a quartic perpetuant, so by superposing any quintic form with the symbolic numbers 43^2 (43^2 being the simplest quartic form which is not a quintic syzygant) we obtain a quintic perpetuant; consider the reduction of the simplest quintic form, viz. $(5)=3.2-32$,
and the operator $V_{-43^2}=(43^2)\,d_0-(53^2)\,d_1+(63^2)\,d_2-\dots;$

operating we find $543^2 = 43^3.2 - 3^4.3 - 43^3 2 + 5(3^5)$, that is 543^2 expressed as a sextic syzygant; now, assuming that this operator will have a similar effect in the case of every reducible quintic form, and not in the case of an irreducible form, it will follow that the generating function for sextic perpetuants is $\frac{x^6}{6} \cdot x^{10} \cdot \frac{x^{15}}{2.3.4.5} = \frac{x^{31}}{2.3.4.5.6}$; the actual result, so far as I have obtained it rigorously, agrees with this.

It seems worth observing that if this principle is sound, the G. F. for n^{ic} perpetuants is $(n > 2)$

$$\frac{x^{2^{n-1}-1}}{2.3.4\ldots n}.$$

Before attacking the general subject of the sextic syzygies it is absolutely necessary to inquire more minutely into the theory of the quintic perpetuants; the main object is to discover the law of occurrence of the exemplar forms in the reduction of the non-exemplar forms. For weight 15, these latter forms are

$$5^3 \;,$$
$$5^2 32,$$

which arise from the fact that the forms 5^2, 532 involve the form 43^2 in their reduction, 43^2 not being a quintic syzygant; in general, every μ^{ic} form will be a non-exemplar perpetuant, which, being decapitated, requires for its reduction a $(\mu - 1)^{\text{ic}}$ perpetuant which is not a μ^{ic} syzygant.

The characteristic of the forms is that their symbols contain each three odd numbers, and it is obvious that every quintic form containing in itself either 5^3 or $5^2 32$ will be a non-exemplar perpetuant, so long as it does not also contain 43^2, when of course it would be exemplar.

Consider now the form of weight w,

$$5^{\kappa} 4^{\lambda} 3^{\mu} 2^{\frac{1}{2}(w-5\kappa-4\lambda-3\mu)},$$

and for brevity denote it by $\widetilde{\kappa\lambda\mu}$; $\widetilde{\kappa\lambda\mu}$ is exemplar if $\lambda \nless 1$ and $\mu \nless 2$; the forms to be considered are three, namely

$$\widetilde{\kappa 0 \mu}\,(\kappa > 1), \quad \widetilde{\kappa\lambda 1}\,(\kappa > 1), \quad \widetilde{\kappa\lambda 0}\,(\kappa > 2);$$

Prof. Cayley has given the law for compounding two symmetric functions into a series of monomials and illustrated it by examples; by thus developing the quintic compounds, the non-exemplar forms we are considering are found to involve, in their reduction, the exemplar forms according to very beautiful laws connected with the binomial coefficients; it will have been observed in Prof. Cayley's paper that he finds that, in the general case, the coefficient of a monomial arising in the expansion is the sum of a number of terms, each of which is a binomial coefficient; the simplicity which is found in the case of perpetuants is therefore rather remarkable. I do not attempt the laborious proof of the following laws in the general case; every number however occurring in the tabulations has been verified for high values of $\varkappa$, λ and μ.

In what follows, those terms involving forms of lower degree are omitted, so that such a result as

$$\overbrace{411} \equiv 2\,\overbrace{134} - \overbrace{144} + \overbrace{223} - \overbrace{233} - \overbrace{322},$$

will not be misunderstood.

All the forms are of the general weight, a number of two's being always supposed present, so as to bring up the weight to w; all the forms of course will not be present when the weight is low, those terms being absent in which the index of the number 2 becomes negative.

The annexed table is formed

	$\overbrace{\theta.1.\kappa+\mu-\theta}$	$\overbrace{\theta.2.\kappa+\mu-\theta}$	$\overbrace{\theta.3.\kappa+\mu-\theta}$	$\overbrace{\theta.4.\kappa+\mu-\theta}$	$\overbrace{\theta.5.\kappa+\mu-\theta}$	$\overbrace{\theta.6.\kappa+\mu-\theta}$	$\overbrace{\theta.7.\kappa+\mu-\theta}$	$\overbrace{\theta.8.\kappa+\mu-\theta}$	$\overbrace{\theta.9.\kappa+\mu-\theta}$	$\overbrace{\theta.10.\kappa+\mu-\theta}$	$\overbrace{\theta.11.\kappa+\mu-\theta}$	⋮	⋮
Row 1	−1											⋮	
“ 2	+1	−1										⋮	
“ 3		+2	−1									⋮	
“ 4		−1	+3	−1								⋮	
“ 5			−3	+4	− 1							⋮	
“ 6			+1	−6	+ 5	− 1						⋮	
“ 7				+4	−10	+ 6	− 1					⋮	
“ 8				−1	+10	−15	+ 7	− 1				⋮	
“ 9					− 5	+20	−21	+ 8	− 1			⋮	
“ 10					+ 1	−15	+35	−28	+ 9	− 1		⋮	
“ 11	. . . .					+ 6	−35	+56	−36	+10	−1	⋮	
	. . . .												

The law is obvious, the column headed $\overbrace{\theta.t.\varkappa+\mu-\theta}$ involving the coefficients in the expansion of $(x+1)^t$, with signs alternately negative and positive. Then $\widetilde{\varkappa 0\mu} \equiv \text{Row } 1\,(\theta=\varkappa-1) + \text{Row } 2\,(\theta=\varkappa-2) + \ldots + \text{Row } \varkappa-1\,(\theta=1)$, or more concisely

$$\widetilde{\varkappa 0\mu} \equiv \sum_{t=\varkappa-1}^{t=1} \text{Row } t\,(\theta=\varkappa-t).$$

As an instance $\widetilde{504} \equiv -\widetilde{415}+\widetilde{316}-\widetilde{326}+2\,\widetilde{227}-\widetilde{237}-\widetilde{128}+3\,\widetilde{138}-\widetilde{148}$.

Again forming a table, viz.,

	$\overbrace{\theta.\lambda+1.2}$	$\overbrace{\theta.\lambda+1.3}$	$\overbrace{\theta.\lambda+2.3}$	$\overbrace{\theta.\lambda+2.4}$	$\overbrace{\theta.\lambda+2.5}$	$\overbrace{\theta.\lambda+3.4}$	$\overbrace{\theta.\lambda+3.5}$	$\overbrace{\theta.\lambda+3.6}$	$\overbrace{\theta.\lambda+3.7}$	$\overbrace{\theta.\lambda+4.5}$	$\overbrace{\theta.\lambda+4.6}$	$\overbrace{\theta.\lambda+4.7}$	$\overbrace{\theta.\lambda+4.8}$	$\overbrace{\theta.\lambda+4.9}$	$\overbrace{\theta.\lambda+5.6}$	$\overbrace{\theta.\lambda+5.7}$	$\overbrace{\theta.\lambda+5.8}$	$\overbrace{\theta.\lambda+5.9}$	$\overbrace{\theta.\lambda+5.10}$	$\overbrace{\theta.\lambda+5.11}$	$\overbrace{\theta.\lambda+6.7}$	. . .
Row 1	−1																					.
" 2		+1	−1																			.
" 3				+2		−1																.
" 4					−1		+3			−1												.
" 5								−3			+4				−1							.
" 6									+1			−6				+5					−1	.
	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

the law being again clear, then

$$\widetilde{\varkappa\lambda 1} \equiv \sum_{t=\varkappa-1}^{t=1} \text{Row } t\,(\theta=\varkappa-t).$$

And again from the table

	$\overbrace{\theta.\lambda+1.2}$	$\overbrace{\theta.\lambda+2.3}$	$\overbrace{\theta.\lambda+2.4}$	$\overbrace{\theta.\lambda+3.4}$	$\overbrace{\theta.\lambda+3.5}$	$\overbrace{\theta.\lambda+3.6}$	$\overbrace{\theta.\lambda+4.5}$	$\overbrace{\theta.\lambda+4.6}$	$\overbrace{\theta.\lambda+4.7}$	$\overbrace{\theta.\lambda+4.8}$	$\overbrace{\theta.\lambda+5.6}$	. . .	. . .
Row 1	+1											.	.
" 2		+1										.	.
" 3			−1	+1								.	.
" 4					−2		+1					.	.
" 5						+1		−3			+1	.	.
	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

which is to be continued on the same principle as the foregoing one to which it is very similar, we have:

$$\widetilde{\kappa\lambda 0} \equiv \sum_{t=\kappa-2}^{t=1} \text{Row } t(\theta = \kappa - t - 1).$$

Thus $$\widetilde{420} \equiv \widetilde{232} + \widetilde{143}.$$

This result may be verified from the previous table by operating with Mr. Hammond's operator d_3 upon the value of $\widetilde{421}$.

Thus $\widetilde{421} \equiv - \widetilde{332} + \widetilde{233} - \widetilde{243} + 2\ \widetilde{144} - \widetilde{154}$

Whence $\widetilde{420} \equiv - \widetilde{331} + \widetilde{232} - \widetilde{242} + 2\ \widetilde{143} - \widetilde{153}$

and $\widetilde{331} \equiv - \widetilde{242} + \widetilde{143} - \widetilde{153}$

Therefore $\widetilde{420} \equiv \widetilde{232} + \widetilde{143}$,

as we before found.

It was absolutely necessary to obtain these theorems in order to form certain sextic syzygies of the general weight as will be done presently.

The sextic syzygies of any weight w are of three kinds: firstly, those formed by capitation of simple identities, which will be alluded to hereafter as simple syzygies; secondly, those formed by capitating the quintic syzygies; thirdly, those formed by capitating the capitatable syzygies of weight $w - 6$.

The first kind, with which for the present we are mostly concerned, arise from any forms, simple or composite, that it is possible to submit to both a 4.2 and a 3.3 capitation; such must be, it is clear, of the form $3^{\kappa}2^{\lambda}.2^{\mu}$, where κ, λ and μ may have any positive, including zero values, consistent with the relation

$$3\kappa + 2(\lambda + \mu) = w - 6;$$

thus from the identity $$3^{\kappa}2^{\lambda}.2^{\mu} = 3^{\kappa}2^{\lambda}.2^{\mu},$$

we get the syzygy $$43^{\kappa}2^{\lambda}.2^{\mu+1} - 3^{\kappa+1}2^{\lambda}.32^{\mu} \equiv,$$

wherein if κ is zero, the resulting quartic form is to be expressed, as it can be at once, in terms of quadric perpetuants.

It will now be proved by considering these simple syzygies in conjunction with another special set of even weight, derived by a second capitation of certain of them, that there can be no sextic perpetuant below weight 31.

It will be convenient to call those syzygies which are derived by a first, second, etc., capitation of simple identities, of the first, second, etc., class.

The special set above referred to will then be of the second class; the syzygies of the first class naturally arrange themselves into groups according to the number of threes occurring in the components of the identities from which they are derived; as will be seen there is only one group in the second class syzygies here discussed. In what follows $\equiv 0$ indicates that the dexter of the syzygy can be expressed in terms of perpetuants of a degree lower than 5, and $\equiv \widetilde{112}$ indicates $\equiv \widetilde{112}$ + lower forms and so on. Thus the first of the following syzygies in its complete form is

$$\begin{array}{rcl} 432^{\kappa}.2 & = & 542^{\kappa} \\ -\,3^2 2^{\kappa}.3 & & -\,53^2 2^{\kappa-1} \\ & & +\,2.(4^2 32^{\kappa-1}) \\ & & +\,(\varkappa+1)\,432^{\kappa+1} \\ & & -\,3.(3^3 2^{\kappa}), \end{array}$$

where on the right-hand side everything is to be rejected. The following values have been thoroughly verified and I think may be relied upon; the syzygies are denoted by the capital letters with suffixes, the suffix denoting the greatest number of threes in any term of the syzygy.

Sextic Syzygies.

Class 1. Group 1. $w = 2\varkappa + 9$.

$$\begin{array}{ll} A_3 & 432^{\kappa}.2 \quad -3^2 2^{\kappa}.3 \qquad \equiv 0, \\ B_3 & 432^{\kappa-1}.2^2 - 3^2 2^{\kappa-1}.32 \equiv \widetilde{112}, \\ C_3 & 432^{\kappa-2}.2^3 - 3^2 2^{\kappa-2}.32^2 \equiv (\varkappa-5)\,\widetilde{112} + \widetilde{122}, \\ D_3 & 432^{\kappa-3}.2^4 - 3^2 2^{\kappa-3}.32^3 \equiv \tfrac{1}{2}(\varkappa^2-11\varkappa+26)\,\widetilde{112} + (\varkappa-7)\,\widetilde{122} + \widetilde{132}, \\ E_3 & 432^{\kappa-4}.2^5 - 3^2 2^{\kappa-4}.32^4 \equiv \tfrac{1}{6}(\varkappa-3)(\varkappa^2-15\varkappa+50)\,\widetilde{112} + \tfrac{1}{2}(\varkappa^2-15\varkappa+52)\,\widetilde{122} \\ & \qquad\qquad + (\varkappa-9)\,\widetilde{132} + \widetilde{142}, \\ & \qquad \text{etc.} \equiv \text{etc.}, \end{array}$$

whence it is easy to infer that every form $\widetilde{1\varkappa2}$ is a sextic syzygant.

Class 1. Group 2. $w = 2\varkappa + 12$.

$$\begin{array}{ll} A_4 & 43^2 2^{\kappa}.2 \quad -3^3 2^{\kappa}.3 \qquad \equiv 0, \\ B_4 & 43^2 2^{\kappa-1}.2^2 - 3^3 2^{\kappa-1}.32 \equiv \widetilde{113}, \\ C_4 & 43^2 2^{\kappa-2}.2^3 - 3^3 2^{\kappa-2}.32^2 \equiv (\varkappa-5)\,\widetilde{113} + 2\ \widetilde{123} + \widetilde{212}, \\ D_4 & 43^2 2^{\kappa-3}.2^4 - 3^3 2^{\kappa-3}.32^3 \equiv \tfrac{1}{2}(\varkappa^2-11\varkappa+26)\,\widetilde{113} + (2\varkappa-13)\,\widetilde{123} \\ & \qquad\qquad + 2\ 133 + (\varkappa-6)\,\widetilde{212} + \widetilde{222}, \\ E_4 & 43^2 2^{\kappa-4}.2^5 - 3^3 2^{\kappa-4}.32^4 \equiv \tfrac{1}{6}(\varkappa-3)(\varkappa^2-15\varkappa+50)\,\widetilde{113} + (\varkappa^2-14\varkappa+45)\,\widetilde{123} \\ & \quad + (2\varkappa-17)\,\widetilde{133} + 2\,\widetilde{143} + \tfrac{1}{2}(\varkappa^2-13\varkappa+38)\,\widetilde{212} + (\varkappa-8)\,\widetilde{222} + \widetilde{232}, \\ & \qquad \text{etc.} \equiv \text{etc.} \end{array}$$

Class 1. Group 3. $w = 2\kappa + 9$.

A_5 $43^3\, 2^{\kappa-3}.\, 2 - 3^4\, 2^{\kappa-3}.\, 3 \equiv \overbrace{112}$,

B_5 $43^3\, 2^{\kappa-4}.\, 2^2 - 3^4\, 2^{\kappa-4}.\, 32 \equiv (\kappa-3)\, \overbrace{112} + \overbrace{114} + \overbrace{122}$,

C_5 $43^3\, 2^{\kappa-5}.\, 2^3 - 3^4\, 2^{\kappa-5}.\, 32^2 \equiv \frac{1}{2}(\kappa-3)(\kappa-4)\, \overbrace{112} + (\kappa-8)\, \overbrace{114} + (\kappa-5)\, \overbrace{122}$
$+ 2\ \overbrace{124} + \overbrace{132} + \overbrace{213}$,

D_5 $43^3\, 2^{\kappa-6}.\, 2^4 - 3^4\, 2^{\kappa-6}.\, 32^3 \equiv \frac{1}{6}(\kappa-3)(\kappa-4)(\kappa-5)\, \overbrace{112} + \frac{1}{2}(\kappa^2 - 17\kappa + 68)\, \overbrace{114}$
$+ \frac{1}{2}(\kappa-5)(\kappa-6)\, \overbrace{122} + 2(\kappa-10)\, \overbrace{124} + (\kappa-7)\, \overbrace{132} + 3\ \overbrace{134} + \overbrace{142}$
$+ (\kappa-9)\, \overbrace{213} + 2\ \overbrace{223} + \overbrace{312}$,

E_5 $43^3\, 2^{\kappa-7}.\, 2^5 - 3^4\, 2^{\kappa-7}.\, 32^4 \equiv \frac{1}{24}(\kappa-3)(\kappa-4)(\kappa-5)(\kappa-6)\, \overbrace{112}$
$+ \frac{1}{6}(\kappa-6)(\kappa-8)(\kappa-13)\, \overbrace{114} + \frac{1}{6}(\kappa-5)(\kappa-6)(\kappa-7)\, \overbrace{122} + (\kappa^2 - 21\kappa + 106)\, \overbrace{124}$
$+ \frac{1}{2}(\kappa-7)(\kappa-8)\, \overbrace{132} + (3\kappa-34)\, \overbrace{134} + (\kappa-9)\, \overbrace{142} + 3\ \overbrace{144} + \overbrace{152}$
$+ \frac{1}{2}(\kappa^2 - 19\kappa + 86)\, \overbrace{213} + (2\kappa-21)\, \overbrace{223} + 2\ \overbrace{233} + (\kappa-10)\, \overbrace{312} + \overbrace{322}$
etc. $\equiv$ etc.

Class 1. Group 4. $w = 2\kappa + 12$.

A_6 $43^4\, 2^{\kappa-3}.\, 2 - 3^5\, 2^{\kappa-3}.\, 3 \equiv \overbrace{113}$,

B_6 $43^4\, 2^{\kappa-4}.\, 2^2 - 3^5\, 2^{\kappa-4}.\, 32 \equiv (\kappa-3)\, \overbrace{113} + \overbrace{115} + 2\ \overbrace{123} + \overbrace{212}$,

C_6 $43^4\, 2^{\kappa-5}.\, 2^3 - 3^5\, 2^{\kappa-5}.\, 32^2 \equiv \frac{1}{2}(\kappa-3)(\kappa-4)\, \overbrace{113} + (\kappa-8)\, \overbrace{115} + (2\kappa-9)\, \overbrace{123}$
$+ 2\ \overbrace{125} + 2\ \overbrace{133} + (\kappa-4)\, \overbrace{212} + \overbrace{214} + \overbrace{222}$,

D_6 $43^4\, 2^{\kappa-6}.\, 2^4 - 3^5\, 2^{\kappa-6}.\, 32^3 \equiv \frac{1}{6}(\kappa-3)(\kappa-4)(\kappa-5)\, \overbrace{113} + \frac{1}{2}(\kappa^2 - 17\kappa + 68)\, \overbrace{115}$
$+ (\kappa-5)^2\, \overbrace{123} + 2(\kappa-10)\, \overbrace{125} + (2\kappa-13)\, \overbrace{133} + 3\ \overbrace{135} + 2\ \overbrace{143}$
$+ \frac{1}{2}(\kappa-4)(\kappa-5)\, \overbrace{212} + (\kappa-9)\, \overbrace{214} + (\kappa-6)\, \overbrace{222} + 2\ \overbrace{224} + \overbrace{232} + \overbrace{313}$,

E_6 $43^4\, 2^{\kappa-7}.\, 2^5 - 3^5\, 2^{\kappa-7}.\, 32^4 \equiv \frac{1}{24}(\kappa-3)(\kappa-4)(\kappa-5)(\kappa-6)\, \overbrace{113}$
$+ \frac{1}{6}(\kappa-6)(\kappa-8)(\kappa-13)\, \overbrace{115} + \frac{1}{6}(\kappa-5)(\kappa-6)(2\kappa-11)\, \overbrace{123}$
$+ (\kappa^2 - 21\kappa + 106)\, \overbrace{125} + (\kappa-7)^2\, \overbrace{133} + 3(\kappa-12)\, \overbrace{135} + (2\kappa-17)\, \overbrace{143}$
$+ 4\ \overbrace{145} + 2\ \overbrace{153} + \frac{1}{6}(\kappa-4)(\kappa-5)(\kappa-6)\, \overbrace{212} + \frac{1}{2}(\kappa^2 - 19\kappa + 86)\, \overbrace{214}$
$+ \frac{1}{2}(\kappa-6)(\kappa-7)\, \overbrace{222} + 2(\kappa-11)\, \overbrace{224} + (\kappa-8)\, \overbrace{232} + 3\ \overbrace{234} + \overbrace{242}$
$+ (\kappa-10)\, \overbrace{313} + 2\ \overbrace{323} + \overbrace{412}$,
etc. $\equiv$ etc.

Class 1. Group 5. $w = 2\kappa + 9$.

A_7 $43^5\, 2^{\kappa-6}.\, 2 - 3^6\, 2^{\kappa-6}.\, 3 \equiv \overbrace{114}$,

B_7 $43^5\, 2^{\kappa-7}.\, 2^2 - 3^6\, 2^{\kappa-7}.\, 32 \equiv (\kappa-6)\, \overbrace{114} + \overbrace{116} + 2\ \overbrace{124} + \overbrace{213}$,

C_7 $43^5\, 2^{\kappa-8}.\, 2^3 - 3^6\, 2^{\kappa-8}.\, 32^2 \equiv \frac{1}{2}(\kappa-6)(\kappa-7)\, \overbrace{114} + (\kappa-11)\, \overbrace{116} + 2(\kappa-8)\, \overbrace{124}$
$+ 2\ \overbrace{126} + 3\ \overbrace{134} + (\kappa-7)\, \overbrace{213} + \overbrace{215} + 2\ \overbrace{223} + \overbrace{312}$,

$$\begin{aligned} D_7\quad 43^5\,2^{\kappa-9}.\,2^4 - 3^6\,2^{\kappa-9}.\,32^3 \equiv{} & \tfrac{1}{6}(\varkappa-6)(\varkappa-7)(\varkappa-8)\,\overbrace{114} + \tfrac{1}{2}(\varkappa^2-23\varkappa+128)\,\overbrace{116} \\ & + (\varkappa-8)(\varkappa-9)\,\overbrace{124} + 2\,(\varkappa-13)\,\overbrace{126} + (3\varkappa-28)\,\overbrace{134} + 3\;\overbrace{136} \\ & + 3\;\overbrace{144} + \tfrac{1}{2}(\varkappa-7)(\varkappa-8)\,\overbrace{213} + (\varkappa-12)\,\overbrace{215} + (2\varkappa-17)\,\overbrace{223} \\ & + 2\;\overbrace{225} + 2\;\overbrace{233} + (\varkappa-8)\,\overbrace{312} + \overbrace{314} + \overbrace{322}, \end{aligned}$$

$$\begin{aligned} E_7\quad 43^5\,2^{\kappa-10}.\,2^5 - 3^6\,2^{\kappa-10}.\,32^4 \equiv{} & \tfrac{1}{24}(\varkappa-6)(\varkappa-7)(\varkappa-8)(\varkappa-9)\,\overbrace{114} \\ & + \tfrac{1}{6}(\varkappa-9)(\varkappa-11)(\varkappa-16)\,\overbrace{116} + \tfrac{1}{3}(\varkappa-8)(\varkappa-9)(\varkappa-10)\,\overbrace{124} \\ & + (\varkappa^2-27\varkappa+178)\,\overbrace{126} + \left\{\tfrac{3}{2}(\varkappa-10)(\varkappa-11) + 2\varkappa-19\right\}\overbrace{134} \\ & + 3\,(\varkappa-15)\,\overbrace{136} + (3\varkappa-34)\,\overbrace{144} + 4\;\overbrace{146} + 3\;\overbrace{154} \\ & + \tfrac{1}{6}(\varkappa-7)(\varkappa-8)(\varkappa-9)\,\overbrace{213} + \tfrac{1}{2}(\varkappa^2-25\varkappa+152)\,\overbrace{215} + (\varkappa-9)^2\,\overbrace{223} \\ & + 2\,(\varkappa-14)\,\overbrace{225} + (2\varkappa-21)\,\overbrace{233} + 3\;\overbrace{235} + 2\;\overbrace{243} + \tfrac{1}{2}(\varkappa-8)(\varkappa-9)\,\overbrace{312} \\ & + (\varkappa-13)\,\overbrace{314} + (\varkappa-10)\,\overbrace{322} + 2\;\overbrace{324} + \overbrace{332} + \overbrace{413}, \end{aligned}$$

$$\text{etc.} \equiv \text{etc.}$$

Class 1. Group 6. $w = 2\varkappa + 12$.

$$A_8\quad 43^6\,2^{\kappa-6}.\,2 - 3^7\,2^{\kappa-6}.\,3 \equiv \overbrace{115},$$

$$B_8\quad 43^6\,2^{\kappa-7}.\,2^2 - 3^7\,2^{\kappa-7}.\,32 \equiv (\varkappa-6)\,\overbrace{115} + \overbrace{117} + 2\;\overbrace{125} + \overbrace{214},$$

$$\begin{aligned} C_8\quad 43^6\,2^{\kappa-8}.\,2^3 - 3^7\,2^{\kappa-8}.\,32^2 \equiv{} & \tfrac{1}{2}(\varkappa-6)(\varkappa-7)\,\overbrace{115} + (\varkappa-11)\,\overbrace{117} + 2\,(\varkappa-8)\,\overbrace{125} \\ & + 2\;\overbrace{127} + 3\;\overbrace{135} + (\varkappa-7)\,\overbrace{214} + \overbrace{216} + 2\;\overbrace{224} + \overbrace{313}, \end{aligned}$$

$$\begin{aligned} D_8\quad 43^6\,2^{\kappa-9}.\,2^4 - 3^7\,2^{\kappa-9}.\,32^3 \equiv{} & \tfrac{1}{6}(\varkappa-6)(\varkappa-7)(\varkappa-8)\,\overbrace{115} + \tfrac{1}{2}(\varkappa^2-23\varkappa+128)\,\overbrace{117} \\ & + (\varkappa-8)(\varkappa-9)\,\overbrace{125} + 2\,(\varkappa-13)\,\overbrace{127} + 3\,(\varkappa-10)\,\overbrace{135} + 3\;\overbrace{137} \\ & + 4\;\overbrace{145} + \tfrac{1}{2}(\varkappa-7)(\varkappa-8)\,\overbrace{214} + (\varkappa-12)\,\overbrace{216} + 2\,(\varkappa-9)\,\overbrace{224} \\ & + 2\;\overbrace{226} + 3\;\overbrace{234} + (\varkappa-8)\,\overbrace{313} + \overbrace{315} + 2\;\overbrace{323} + \overbrace{412}, \end{aligned}$$

$$\begin{aligned} E_8\quad 43^6\,2^{\kappa-10}.\,2^5 - 3^7\,2^{\kappa-10}.\,32^4 \equiv{} & \tfrac{1}{24}(\varkappa-6)(\varkappa-7)(\varkappa-8)(\varkappa-9)\,\overbrace{115} \\ & + \tfrac{1}{6}(\varkappa-9)(\varkappa-11)(\varkappa-16)\,\overbrace{117} + \tfrac{1}{3}(\varkappa-8)(\varkappa-9)(\varkappa-10)\,\overbrace{125} \\ & + (\varkappa^2-27\varkappa+178)\,\overbrace{127} + \left\{\tfrac{3}{2}(\varkappa-10)(\varkappa-11) - 1\right\}\overbrace{135} + 3\,(\varkappa-15)\,\overbrace{137} \\ & + (4\varkappa-45)\,\overbrace{145} + 4\;\overbrace{147} + 4\;\overbrace{155} + \tfrac{1}{6}(\varkappa-7)(\varkappa-8)(\varkappa-9)\,\overbrace{214} \\ & + \tfrac{1}{2}(\varkappa^2-25\varkappa+152)\,\overbrace{216} + (\varkappa-9)(\varkappa-10)\,\overbrace{224} + 2\,(\varkappa-14)\,\overbrace{226} \\ & + (3\varkappa-31)\,\overbrace{234} + 2\;\overbrace{236} + 3\;\overbrace{244} + \tfrac{1}{2}(\varkappa-8)(\varkappa-9)\,\overbrace{313} + (\varkappa-13)\,\overbrace{315} \\ & + (2\varkappa-19)\,\overbrace{323} + 2\;\overbrace{325} + 2\;\overbrace{333} + (\varkappa-9)\,\overbrace{412} + \overbrace{414} + \overbrace{422}, \end{aligned}$$

$$\text{etc.} \equiv \text{etc.}$$

Class 1. Group 7. $w = 2\varkappa + 9$.

A_9 $43^7 2^{\kappa-9}.2 - 3^8 2^{\kappa-9}.3 \equiv \widetilde{116}$,

B_9 $43^7 2^{\kappa-10}.2^2 - 3^8 2^{\kappa-10}.32 \equiv (\varkappa - 9)\widetilde{116} + \widetilde{118} + 2\,\widetilde{126} + \widetilde{215}$,

C_9 $43^7 2^{\kappa-11}.2^3 - 3^8 2^{\kappa-11}.32^2 \equiv \tfrac{1}{2}(\varkappa - 9)(\varkappa - 10)\widetilde{116} + (\varkappa - 14)\widetilde{118}$
$+ 2(\varkappa - 11)\widetilde{126} + 2\,\widetilde{128} + 3\,\widetilde{136} + (\varkappa - 10)\widetilde{215} + \widetilde{217} + 2\,\widetilde{225} + \widetilde{314}$,

D_9 $43^7 2^{\kappa-12}.2^4 - 3^8 2^{\kappa-12}.32^3 \equiv \tfrac{1}{6}(\varkappa - 9)(\varkappa - 10)(\varkappa - 11)\widetilde{116}$
$+ \tfrac{1}{2}(\varkappa^2 - 29\varkappa + 206)\widetilde{118} + (\varkappa - 11)(\varkappa - 12)\widetilde{126} + 2(\varkappa - 16)\widetilde{128}$
$+ 3(\varkappa - 13)\widetilde{136} + 3\,\widetilde{138} + 4\,\widetilde{146} + \tfrac{1}{2}(\varkappa - 10)(\varkappa - 11)\widetilde{215}$
$+ (\varkappa - 15)\widetilde{217} + 2(\varkappa - 12)\widetilde{225} + 2\,\widetilde{227} + 3\,\widetilde{235} + (\varkappa - 11)\widetilde{314}$
$+ \widetilde{316} + 2\,\widetilde{324} + \widetilde{413}$,

E_9 $43^7 2^{\kappa-13}.2^5 - 3^8 2^{\kappa-13}.32^4 \equiv \tfrac{1}{24}(\varkappa - 9)(\varkappa - 10)(\varkappa - 11)(\varkappa - 12)\widetilde{116}$
$+ \tfrac{1}{6}(\varkappa - 12)(\varkappa - 14)(\varkappa - 19)\widetilde{118} + \tfrac{1}{3}(\varkappa - 11)(\varkappa - 12)(\varkappa - 13)\widetilde{126}$
$+ (\varkappa^2 - 33\varkappa + 268)\widetilde{128} + \tfrac{3}{2}(\varkappa - 13)(\varkappa - 14)\widetilde{136} + 3(\varkappa - 18)\widetilde{138}$
$+ 4(\varkappa - 15)\widetilde{146} + 4\,\widetilde{148} + 5\,\widetilde{156} + \tfrac{1}{6}(\varkappa - 10)(\varkappa - 11)(\varkappa - 12)\widetilde{215}$
$+ \tfrac{1}{2}(\varkappa^2 - 31\varkappa + 236)\widetilde{217} + (\varkappa - 12)(\varkappa - 13)\widetilde{225} + 2(\varkappa - 17)\widetilde{227}$
$+ 3(\varkappa - 14)\widetilde{235} + 3\,\widetilde{237} + 4\,\widetilde{245} + \tfrac{1}{2}(\varkappa - 11)(\varkappa - 12)\widetilde{314}$
$+ (\varkappa - 16)\widetilde{316} + 2(\varkappa - 13)\widetilde{324} + 2\,\widetilde{326} + 3\,\widetilde{334} + (\varkappa - 12)\widetilde{413}$
$+ \widetilde{415} + 2\,\widetilde{423} + \widetilde{512}$,

etc. $\equiv$ etc.

It will be observed that the A, B, C, D, E syzygies attain their final forms in groups 3, 4, 5, 6, 7 respectively, and generally the N syzygies reach their final form in group $n + 2$; a general formula can be obtained for the N syzygies for group $n + 2$ and succeeding groups; it is evident on inspection that the number of terms in a final form is a sum of odd numbers, proceeding regularly from unity; thus in the case of the C syzygies there is one term of the form $\widetilde{3\lambda\mu}$, three of $\widetilde{2\lambda\mu}$, five of $\widetilde{1\lambda\mu}$, giving a total of nine terms; the numbers of terms in the final forms of the A, B, C . . . syzygies are therefore the successive square numbers 1, 4, 9 . . . . After the final forms have been reached, the syzygies A_x, B_x, C_x, D_x, E_x . . . are derived from the syzygies A_{x-2}, B_{x-2}, C_{x-2}, D_{x-2}, E_{x-2}, . . . by substituting therein $\varkappa - 3$ for $\varkappa$ and increasing the third number under the symbol $\widetilde{\quad}$ by unity.

In Group 7 all the syzygies written down are in their final forms; what is apparently the law of the coefficients in the A, B, C, D, E syzygies may be shown by the following scheme. Consider a group λ, of weight $3\lambda + 2\mu + 6$, commencing with the syzygy

$$A_{\lambda+2}\ 43^{\lambda}2^{\mu}.2 - 3^{\lambda+1}2^{\mu}.3 \equiv \widetilde{1.1.\lambda - 1},$$

any number in the table represents the coefficient of the perpetuant in the same row and in the left-hand column, in the syzygy denoted by the capital letter with suffix, at the head of its column.

	$A_{\lambda+2}$	$B_{\lambda+2}$	$C_{\lambda+2}$	$D_{\lambda+2}$	$E_{\lambda+2}$
$\overbrace{1 . 1 . \lambda-1}$	1	μ	$\frac{1}{2}\mu(\mu-1)$	$\frac{1}{6}\mu(\mu-1)(\mu-2)$	$\frac{1}{24}\mu(\mu-1)(\mu-2)(\mu-3)$
$\overbrace{1 . 2 . \lambda-1}$		2	$2(\mu-2)$	$\frac{2}{2}(\mu-2)(\mu-3)$	$\frac{2}{6}(\mu-2)(\mu-3)(\mu-4)$
$\overbrace{1 . 3 . \lambda-1}$			3	$3(\mu-4)$	$\frac{3}{2}(\mu-4)(\mu-5)$
$\overbrace{1 . 4 . \lambda-1}$				4	$4(\mu-6)$
$\overbrace{1 . 5 . \lambda-1}$					5
$\overbrace{1 . 1 . \lambda+1}$		1	$\mu-5$	$\frac{1}{2}(\mu-5)(\mu-6)-2$	$\frac{1}{6}(\mu-5)(\mu-6)(\mu-7)-2(\mu-5)$
$\overbrace{1 . 2 . \lambda+1}$			2	$2(\mu-7)$	$\frac{2}{2}(\mu-7)(\mu-8)-4$
$\overbrace{1 . 3 . \lambda+1}$				3	$3(\mu-9)$
$\overbrace{1 . 4 . \lambda+1}$					4
$\overbrace{2 . 1 . \lambda-2}$		1	$\mu-1$	$\frac{1}{2}(\mu-1)(\mu-2)$	$\frac{1}{6}(\mu-1)(\mu-2)(\mu-3)$
$\overbrace{2 . 2 . \lambda-2}$			2	$2(\mu-3)$	$\frac{2}{2}(\mu-3)(\mu-4)$
$\overbrace{2 . 3 . \lambda-2}$				3	$3(\mu-5)$
$\overbrace{2 . 4 . \lambda-2}$					4
$\overbrace{2 . 1 . \lambda}$			1	$\mu-6$	$\frac{1}{2}(\mu-6)(\mu-7)-2$
$\overbrace{2 . 2 . \lambda}$				2	$2(\mu-8)$
$\overbrace{2 . 3 . \lambda}$					3
$\overbrace{3 . 1 . \lambda-3}$			1	$\mu-2$	$\frac{1}{2}(\mu-2)(\mu-3)$
$\overbrace{3 . 2 . \lambda-3}$				2	$2(\mu-4)$
$\overbrace{3 . 3 . \lambda-3}$					3
$\overbrace{3 . 1 . \lambda-1}$				1	$\mu-7$
$\overbrace{3 . 2 . \lambda-1}$					2
$\overbrace{4 . 1 . \lambda-4}$				1	$\mu-3$
$\overbrace{4 . 2 . \lambda-4}$					2
$\overbrace{4 . 1 . \lambda-2}$					1
$\overbrace{5 . 1 . \lambda-5}$					1

From this it appears that the coefficients in any block are derived from those in the preceding one, by shifting them one column to the right and writing $\mu - 1$ for μ; further in any one block, the coefficients in the second part of the block are derived from those in the first part by shifting them one column to the right and writing $\mu - 5$ for μ; at the same time subtracting from them twice the column to the left, next to it but one, with $\mu - 5$ written for μ therein.

It only remains to examine the first part of the first block; the law of the first row therein is evident; the t^{th} row is formed from the first row by multiplying it by t, writing $\mu - 2(t-1)$ for μ and shifting it $t - 1$ columns to the right.

The complete law thus appears to be defined, but I have attempted no proof of it.

Passing now to the special group of syzygies of Class 2, before referred to; they arise from the simple identity $2^{\kappa-\lambda}.2^{\lambda} = 2^{\kappa-\lambda}.2^{\lambda}$; a first capitation gives the Class 1 syzygies included in $32^{\kappa-\lambda}.32^{\lambda} - 42^{\kappa-\lambda}.2^{\lambda+1} \equiv$; in which the component $42^{\kappa-\lambda}$ is to be at once expressed in terms of quadric perpetuants; the dexter of this syzygy cannot involve quintic perpetuants, because the sinister contains no term with three odd symbolic numbers; the whole of these syzygies can therefore be at once again capitated as they are; and since the dexter of the above written syzygy consists entirely of quintic compounds with two odd numbers, quartic and cubic perpetuants with two odd numbers, together with terms consisting wholly of even numbers, and since the first three species of terms can be submitted to a 4.2 capitation, an operation which does not increase the number of odd numbers which they contain, it easily follows that the only term which need be considered after the second capitation is $3^2 2^{\kappa-\lambda}.3^2 2^{\lambda}$; this is part of a theory of abbreviation which is more fully entered upon afterwards. We have then:

Class 2. Special Group. $w = 2\varkappa + 12$.

$A'_4 \quad 3^2 2^{\kappa}.3^2 \quad - \ldots \equiv - , \ \widetilde{113}$

$B'_4 \quad 3^2 2^{\kappa-1}.3^2 2 - \ldots \equiv -(\varkappa - 4)\,\widetilde{113} - 3\ \widetilde{123} - 2\ \widetilde{212},$

$C'_4 \quad 3^2 2^{\kappa-2}.3^2 2^2 - \ldots \equiv -\{\tfrac{1}{2}(\varkappa - 4)(\varkappa - 5) - 2\}\widetilde{113} - 3(\varkappa - 6)\,\widetilde{123} - 3\ \widetilde{133}$

$\qquad - 2(\varkappa - 6)\,\widetilde{212} - 2\ \widetilde{222},$

$D'_4 \quad 3^2 2^{\kappa-3}.3^2 2^3 - \ldots \equiv -\{\tfrac{1}{6}(\varkappa - 4)(\varkappa - 5)(\varkappa - 6) - 2(\varkappa - 4)\}\widetilde{113}$

$\qquad - 3\{\tfrac{1}{2}(\varkappa - 6)(\varkappa - 7) - 2\}\ \widetilde{123} - 3(\varkappa - 8)\,\widetilde{133} - 3\ \widetilde{143}$

$\qquad - 2\{\tfrac{1}{2}(\varkappa - 6)(\varkappa - 7) - 2\}\ \widetilde{212} - 2(\varkappa - 8)\,\widetilde{222} - 2\ \widetilde{232}.$

etc. $\equiv$ etc.

Whence forming a scheme as before,

	A'_4	B'_4	C_4'	D'_4
$\overbrace{113}$	-1	$-(\kappa-4)$	$-\{\frac{1}{2}(\kappa-4)(\kappa-5)-2\}$	$-\{\frac{1}{6}(\kappa-4)(\kappa-5)(\kappa-6)-2(\kappa-4)\}$
$\overbrace{123}$		-3	$-3(\kappa-6)$	$-3\{\frac{1}{2}(\kappa-6)(\kappa-7)-2\}$
$\overbrace{133}$			-3	$-3(\kappa-8)$
$\overbrace{143}$				-3

$\overbrace{212}$		-2	$-2(\kappa-6)$	$-2\{\frac{1}{2}(\kappa-6)(\kappa-7)-2\}$
$\overbrace{222}$			-2	$-2(\kappa-8)$
$\overbrace{232}$				-2

and the law is apparent.

Proceeding to examine the groups of syzygies, it is seen that $\overbrace{11\mu}$ is a sextic syzygant, and as remarked before, Group 1 shows that $\overbrace{1\lambda 2}$ is so also; from C_4 and B_4' we see that so are $\overbrace{123}$ and $\overbrace{212}$; from D_4 and C_4', $\overbrace{133}$ and $\overbrace{222}$ are so also, and so on, combining the members of the second and special groups in pairs it is seen that every quintic perpetuant with four odd numbers is a sextic syzygant; those containing a lesser number have been shown to be so also. If the attempt is now made to express either of the forms $\overbrace{312}$, $\overbrace{213}$, $\overbrace{124}$ in a similar manner, it is found that the coincidence of the coefficients involved presents an insuperable obstruction; by making a list of the forms occurring for different weights, commencing with the weight 15, it will be seen to what point an advance has been made. We have

$w=$							
15	$\overbrace{112}$.						
17	$\overbrace{112}$.						
18	$\overbrace{113}$.						
19	$\overbrace{112}$,	$\overbrace{122}$.					
20	$\overbrace{113}$,	$\overbrace{212}$.					
21	$\overbrace{112}$,	$\overbrace{114}$,	$\overbrace{122}$.				
22	$\overbrace{113}$,	$\overbrace{123}$,	$\overbrace{212}$.				
23	$\overbrace{112}$,	$\overbrace{114}$,	$\overbrace{122}$,	$\overbrace{132}$,	$\overbrace{213}$.		
24	$\overbrace{113}$,	$\overbrace{115}$,	$\overbrace{123}$,	$\overbrace{212}$,	$\overbrace{222}$.		
25	$\overbrace{112}$,	$\overbrace{114}$,	$\overbrace{122}$,	$\overbrace{132}$,	$\overbrace{124}$,	$\overbrace{213}$,	$\overbrace{312}$.
26	$\overbrace{113}$,	$\overbrace{115}$,	$\overbrace{123}$,	$\overbrace{133}$,	$\overbrace{212}$,	$\overbrace{222}$,	$\overbrace{214}$,

so that certainly every form of a lower weight than 23 is a sextic syzygant; since the form $\widetilde{124}$ does not occur for weight 23, it is evident from B_7 or C_5 that for weight 23, $\widetilde{213}$ is a syzygant; all forms of weight 24 are certainly syzygants and from B_8 so are those of weight 26.

The conclusion arrived at is that the only possible weight, below weight 33, for which there can be a sextic perpetuant is 31.

From the groups of simple syzygies of Class 1 are derivable two series of capitatable syzygies; thus for an odd weight:

$$\begin{aligned}
&A_3 \equiv 0, \\
&B_3 - A_5 \equiv 0, \\
&2B_3 + C_3 - B_5 + A_7 \equiv 0, \\
&3B_3 + 2C_3 + D_3 - C_5 - 2A_7 + B_7 - A_9 \equiv 0, \\
&4B_3 + 3C_3 + 2D_3 + E_3 - D_5 + A_7 - 2B_7 + C_7 + 4A_9 - B_9 + A_{11} \equiv 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

These may also be written

$$\begin{aligned}
&A_3 \equiv 0, \\
&B_3 - A_5 \equiv 0, \\
&2B_3 + C_3 - (B_5 - A_7) \equiv 0, \\
&3B_3 + 2C_3 + D_3 - \{2B_5 + C_5 - (B_7 - A_9)\} + 2(B_5 - A_7) \equiv 0, \\
&4B_3 + 3C_3 + 2D_3 + E_3 - [3B_5 + 2C_5 + D_5 - \{2B_7 + C_7 - (B_9 - A_{11})\} + 2(B_7 - A_9)] \\
&\qquad + 2\{2B_5 + C_5 - (B_7 - A_9)\} - (B_5 - A_7) \equiv 0, \\
&\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

but the law does not seem to be clear.

Making a unit increase of suffixes throughout, a similar series is obtained for each even weight.

It is now necessary to examine the remaining syzygies of weight 25, in order to discover if the forms $\widetilde{312}$, $\widetilde{213}$, $\widetilde{124}$ of that weight are separately expressible as sextic syzygants; for this purpose, there is no need to attend to those syzygies which are derived by a sextic capitation of quintic syzygies; for consider the syzygy $32.2 - 3.2^2 = 43 + 32^2$; this is capable of 4.2 capitation throughout, and it is plain that it can be operated upon an infinite number of times in a similar way, without there being any necessity to introduce another odd number into any term; similarly the whole of the quintic syzygies admit of infinite 4.2 capitation and can never involve a quintic perpetuant. It is only requisite to consider then those that are derived from the sextic syzygies of weight 19. We first require the syzygies of weight 13; these are

$$\begin{array}{llll}
3^22.32 & 432^2.2 & 432.2^2 & 43.2^3 \\
-3^2.32^2 & -3^22^2.3 & -3^22.32 & -3^2.32^2 \\
-4^23.2 & -32^3.2^2 & -2(32^2.2^3) & -32^3.2^2 \\
-3^22^2.3 & +4(32^4.2) & +2(32^3.2^2) & +4(32^4.2) \\
-3^3.2^2 & +3^32.2 & +4(32^4.2) & +3^3.2^2 \\
+3^32.2 & -4^232 & +2(3^3.2^2) & +3^32.2 \\
+32^2.2^3 & -43^3 & +3^32.2 & +4^232 \\
-2(32^3.2^2) & -4(432^3) & -2(4^232) & -43^3 \\
+2(32^4.2) & +3^32^2 & -7(432^3) & -2(432^3) \\
+2(4^232) & -10(32^5)=0 & +2(43^3) & -10(32^5)=0 \\
-4(43^3) & & +2(3^32^2) & \\
+432^3 & & -20(32^5)=0 & \\
-3^32^2=0 & & &
\end{array}$$

Capitating these, and placing the simple syzygies first, we have the following fifteen syzygies of weight 19; the $\varkappa$ forms being those derived from quintic syzygies.

$$\begin{array}{llll}
A_3=432^5.2 & B_3=432^4.2^2 & C_3=432^3.2^3 & D_3=432^2.2^4 \\
\quad -3^22^5.3 & \quad -3^22^4.32 & \quad -3^22^3.32^2 & \quad -3^22^2.32^3 \\
\quad \equiv 0 & \quad \equiv \overbrace{112} & \quad \equiv \overbrace{122} & \quad \equiv -2\,\overbrace{112}-2\,\overbrace{122} \\
E_3=432.2^5 & F_3=43.2^6 & & \\
\quad -3^22.32^4 & \quad -3^2.32^5 & & \\
\quad \equiv \overbrace{122} & \quad \equiv \overbrace{112} & & \\
A_5=43^32^2.2 & B_5=43^32.2^2 & C_5=43^3.2^3 & \\
\quad -3^42^2.3 & \quad -3^42.32 & \quad -3^4.32^2 & \\
\quad \equiv \overbrace{112} & \quad \equiv \overbrace{122}+2\,\overbrace{112} & \quad \equiv \overbrace{112} & \\
\varkappa_1=432^4.2^2 & \varkappa'_1=432^3.2^3 & & \\
\quad -43.2^6 & \quad -432.2^5 & & \\
\quad -4^232^3.2 & \quad -4^332.2 & & \\
\quad -4(432^5.2) & \quad -2(4^232^3.2) & & \\
\quad \equiv 0 & \quad -5(432^5.2) & & \\
 & \quad \equiv 0 & &
\end{array}$$

$$\begin{aligned}
A'_5 \equiv\ & 3^32.3^22 \\
& - 3^3.3^22^2 \\
& - 4^33.2^2 \\
& - 3^32^2.3^2 \\
& - 43^3.2^3 \\
& + 43^32.2^2 \\
& + 432^2.2^4 \\
& - 2(432^3.2^3) \\
& + 2(432^4.2^2) \\
& + 2(4^332.2) \\
& - 4(4^23^3.2) \\
& + 4^232^3.2 \\
& - 43^32^2.2 \\
& \equiv - 3\ \overbrace{122}
\end{aligned}$$

$$\begin{aligned}
B'_5 \equiv\ & 4^232^2.2^2 \\
& - 3^32^2.3^2 \\
& - 432^3.2^3 \\
& + 4(432^4.2^2) \\
& + 43^32.2^2 \\
& - 4^332.2 \\
& - 4^23^3.2 \\
& - 4(4^232^3.2) \\
& + 43^32^2.2 \\
& - 10(432^5.2) \\
& \equiv 3\ \overbrace{112}
\end{aligned}$$

$$\begin{aligned}
C_5' \equiv\ & 4^232.2^3 \\
& - 3^32.3^22 \\
& - 2(432^2.2^4) \\
& + 2(432^3.2^3) \\
& + 4(432^4.2^2) \\
& + 2(43^3.2^3) \\
& + 43^32.2^2 \\
& - 2(4^332.2) \\
& - 7(4^232^3.2) \\
& + 2(4^23^3.2) \\
& + 2(43^32^2.2) \\
& - 20(432^5.2) \\
& \equiv 3\ \overbrace{122} + 6\ \overbrace{112}
\end{aligned}$$

$$\begin{aligned}
D'_5 \equiv\ & 4^23.2^4 \\
& - 3^3.3^22^2 \\
& - 432^3.2^3 \\
& + 4(432^4.2^2) \\
& + 43^3.2^3 \\
& + 43^32.2^2 \\
& + 4^332.2 \\
& - 4^23^3.2 \\
& - 2(4^232^3.2) \\
& - 10(432^5.2) \\
& \equiv 3\ \overbrace{112}
\end{aligned}$$

which may be verified by the relations:

$$\begin{aligned}
& 4^23^3.2 \equiv - 3^22^3.32^2 \equiv - 3^22.32^4 \equiv \overbrace{122}, \\
& 43^32^2.2 \equiv 43^3.2^3 \equiv - 3^22^4.32 \equiv - 3^2.32^5 \equiv \overbrace{112}, \\
& 43^32.2^2 \equiv 2\ \overbrace{112} + \overbrace{122}, \\
& 3^22^2.32^3 \equiv 2\ \overbrace{112} + 2\ \overbrace{122},
\end{aligned}$$

every other being a null form.

From the foregoing is obtained the capitatable series of thirteen syzygies, viz:

$$\begin{array}{llll}
A_3 \equiv 0, & A_5 - B_3 \equiv 0, & \varkappa_1 \equiv 0, & A'_5 + 3C_3 \equiv 0, \\
D_3 + 2C_3 + 2B_3 \equiv 0, & B_5 - 2B_3 - C_3 \equiv 0, & \varkappa'_1 \equiv 0, & B'_5 - 3B_3 \equiv 0, \\
E_3 - C_3 \equiv 0, & C_5 - B_3 \equiv 0, & & C_5' - 3C_3 - 6B_3 \equiv 0, \\
F_3 - B_3 \equiv 0, & & & D'_5 - 3B_3 \equiv 0.
\end{array}$$

Before proceeding it is convenient to make a few remarks in order to shorten the remaining work.

Reflection will show that we can at this stage neglect a large number of terms as having no influence on the present investigation; the capitatable syzygies above written down are more strictly what Prof. Cayley terms Congruences, which have to be completed into full syzygies prior to capitation; now it is clear that any term of the congruence, which contains but one part three, will only give rise to such terms when it is completed, and that it and its dependent terms are capable of 4.2 capitation, which will produce no term that can give rise to a quintic perpetuant; for the present purpose therefore all terms containing but one three are at once neglected.

Bearing in mind that we are only concerned with the forms $\widetilde{312}$, $\widetilde{213}$, $\widetilde{124}$, which contain five odd numbers, it having been shown that the remainder are sextic syzygants, another great abbreviation may be employed; for any term containing three threes will give rise to terms containing three threes capable of 4.2 capitation; an operation which cannot produce perpetuants with five odd numbers; it follows therefore that every 4.2 form containing three threes may be at once neglected, and that every 3.3 form containing three threes, at most, may be capitated as it stands, *i. e.* without completion.

The only terms then that need completion are the 3.3 terms containing five threes. We have

$$\begin{aligned} 3^42^2.3 &\equiv 53^42 + 5(3^52^2), \\ 3^52.2 &= 53^42 + 43^5 + 2(3^52^2), \end{aligned}$$

thus

$$3^42^2.3 \equiv 3^52.2 - 43^5 + 3(3^52^2);$$

and

$$\begin{aligned} 3^42.32 &\equiv 2(3^5.2^2) + 3(3^52.2) + 2(43^5) + 2(3^52^2), \\ 3^4.32^2 &\equiv 3(3^5.2^2) + 3^52.2 - 43^5, \\ 3^3.3^22^2 &\equiv 3(3^5.2^2) + 3(3^52.2) - 3(43^5) + 3^52^2, \\ 3^32.3^22 &\equiv 6(3^5.2^2) + 4(3^52.2) + 6(43^5) + 6(3^52^2), \\ 3^32^2.3^2 &\equiv 3^5.2^2 + 3(3^52.2) - 3(43^5) + 3(3^52^2). \end{aligned}$$

Using these results in completing and then capitating, we obtain the following thirteen syzygies, in which only those terms which are essential to our purpose are retained.

$$\begin{array}{lllll} \text{No. 1.} & \text{No. 2.} & \text{No. 3.} & \text{No. 4.} & \text{No. 5.} \\ \quad\vdots & \quad\vdots & \quad\vdots & \quad\vdots & \quad\vdots \\ + 3^32^5.3^2 & - 3^32^2.3^22^3 & - 3^32.3^22^4 & - 3^3.3^22^5 & - 3^52^2.3^2 \\ \quad\equiv & - 2(3^32^3.3^22^2) & + 3^32^3.3^22^2 & + 3^32^4.3^22 & + 43^52.2^2 \\ & - 2(3^32^4.3^22) & \quad\equiv & \quad\equiv & - 4^23^5.2 \\ & \quad\equiv & & & + 3(43^52^2.2) \\ & & & & + 3^32^4.3^22 \\ & & & & \quad\equiv \end{array}$$

No. 6.

$$\begin{array}{l} \vdots \\ -\,3^{5}2\,.\,3^{2}2 \\ +\,2\,(43^{5}.\,2^{3}) \\ +\,3\,(43^{5}2\,.\,2^{2}) \\ +\,2\,(4^{2}3^{5}.\,2) \\ +\,2\,(43^{5}2^{2}.\,2) \\ +\,2\,(3^{3}2^{4}.\,3^{2}2) \\ +\,3^{3}2^{3}.\,3^{2}2^{2} \\ \equiv \end{array}$$

No. 7.

$$\begin{array}{l} \vdots \\ -\,3^{5}.\,3^{2}2^{2} \\ +\,3\,(43^{5}.\,2^{3}) \\ +\,43^{5}2\,.\,2^{2} \\ -\,4^{2}3^{5}.\,2 \\ +\,3^{3}2^{4}.\,3^{2}2 \\ \equiv \end{array}$$

No. 8.

$$\begin{array}{l} \vdots \\ \equiv 0 \end{array}$$

No. 9.

$$\begin{array}{l} \vdots \\ \equiv 0 \end{array}$$

No. 10.

$$\begin{array}{l} \vdots \\ 3^{4}2\,.\,3^{3}2 \\ -\,2\,(43^{5}.\,2^{3}) \\ +\,2(43^{5}2\,.\,2^{2}) \\ -\,12\,(4^{2}3^{5}.\,2) \\ -\,2\,(43^{5}2^{2}.\,2) \\ -\,3\,(3^{3}2^{3}.3^{2}2^{2}) \\ \equiv \end{array}$$

No. 11.

$$\begin{array}{l} \vdots \\ -\,3^{4}2^{2}.\,3^{3} \\ +\,43^{5}.\,2^{3} \\ +\,3\,(43^{5}2\,.\,2^{2}) \\ -\,3\,(4^{2}3^{5}.\,2) \\ +\,3\,(43^{5}2^{2}.\,2) \\ +\,3\,(3^{3}2^{4}.\,3^{2}2) \\ \equiv \end{array}$$

No. 12.

$$\begin{array}{l} \vdots \\ -\,3^{4}2\,.\,3^{3}2 \\ +\,6\,(43^{5}.\,2^{3}) \\ +\,4\,(43^{5}2\,.\,2^{2}) \\ +\,6\,(4^{2}3^{5}.\,2) \\ +\,6\,(43^{5}2^{2}.\,2) \\ +\,3\,(3^{3}2^{3}.\,3^{2}2^{2}) \\ +\,6\,(3^{3}2^{4}.\,3^{2}2) \\ \equiv \end{array}$$

No. 13.

$$\begin{array}{l} \vdots \\ -\,3^{4}.\,3^{3}2^{2} \\ +\,3\,(43^{5}.\,2^{3}) \\ +\,3\,(43^{5}2\,.\,2^{2}) \\ -\,3\,(4^{2}3^{5}.\,2) \\ +\,43^{5}2^{2}.\,2 \\ +\,3\,(3^{3}2^{4}.\,3^{2}2) \\ \equiv \end{array}$$

The end of this long investigation has now been nearly reached, and, by reason of the extreme peculiarity of the result presently obtained, great care has been taken to afford a means of checking the work at each stage.

The following values are now required; only the terms $\widetilde{312}$, $\widetilde{213}$, $\widetilde{124}$, being retained.

$$\begin{array}{l} 3^{3}2^{5}.\,3^{2} \equiv 0, \\ 3^{3}2^{4}.\,3^{2}2 \equiv -\,2\ \widetilde{213} - 3\ \widetilde{124}, \\ 3^{3}2^{3}.\,3^{2}2^{2} \equiv -\,3\ \widetilde{312} + \widetilde{213} + 4\ \widetilde{124}, \\ 3^{3}2^{2}.\,3^{2}2^{3} \equiv +\,6\ \widetilde{312} + 3\ \widetilde{213}, \\ 3^{3}2\,.\,3^{2}2^{4} \equiv -\,3\ \widetilde{312} - \widetilde{213}, \\ 3^{3}.\,3^{2}2^{5} \equiv -\,\widetilde{213} - \widetilde{124}, \\ 4^{2}3^{5}.\,2 \equiv +\,\widetilde{124}. \end{array}$$

$$\begin{array}{l} 43^{5}2^{2}.\,2 \equiv 0, \\ 43^{5}2\,.\,2^{2} \equiv +\,\widetilde{213} + 2\ \widetilde{124}, \\ 43^{5}.\,2^{3} \equiv +\,\widetilde{312} + \widetilde{213}, \\ 3^{4}2^{2}.\,3^{3} \equiv 0, \\ 3^{4}2\,.\,3^{3}2 \equiv 0, \\ 3^{4}2^{2}.\,3^{3}2^{2} \equiv 0, \end{array}$$

Substituting these values in the above written syzygies, we find:

$$
\begin{aligned}
&\text{No. } 1 \equiv 0,\\
&\text{No. } 2 \equiv -(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 3 \equiv +2\,(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 4 \equiv -(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 5 \equiv -(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 6 \equiv -(\overbrace{312} - 2\,\overbrace{124}) + 2(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 7 \equiv +3(\overbrace{312} - 2\,\overbrace{124}) + 2(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 8 \equiv 0,\\
&\text{No. } 9 \equiv 0,\\
&\text{No. } 10 \equiv +7(\overbrace{312} - 2\,\overbrace{124}) - 3(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 11 \equiv +(\overbrace{312} - 2\,\overbrace{124}) - 2(\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 12 \equiv -3(\overbrace{312} - 2\,\overbrace{124}) + (\overbrace{213} + 2\,\overbrace{124}),\\
&\text{No. } 13 \equiv +3(\overbrace{312} - 2\,\overbrace{124}),
\end{aligned}
$$

from which it appears that, although we have

$$\overbrace{312} - 2\,\overbrace{124} = \text{sextic syzygant},$$

$$\overbrace{213} + 2\,\overbrace{124} = \text{sextic syzygant},$$

yet it is impossible to express $\overbrace{312}$, $\overbrace{213}$, $\overbrace{124}$ each separately as a sextic syzygant. This means that there is a sextic perpetuant of the weight 31.

The extraordinary character of this result consists in the fact that there are altogether no fewer than sixteen syzygies involving two or all of the three forms, and that yet they are so locked together that the elimination of two of them is impossible.

Interpreting the symbols, it is found that there are three sextic perpetuants, so far of weight 31, viz.: $65^3 43^2$, $65^2 43^3 2$, $654^2 3^4$, of which the last is obviously the exemplar form, the first two being non-exemplar.

The only other possible non-exemplar forms are $6^2 5^2 3^3$, $6^2 53^4 2$, $6^2 43^5$, $6^2 3^5 2^2$, the first three of which are not, when decapitated, reducible without the aid of the form $54^2 3^4$. The last is so.

We have therefore for weight 31

$$1 \text{ exemplar form } 654^2 3^4$$

$$5 \text{ non-exemplar forms } \begin{cases} 6^2 5^2 3^3 \\ 6^2 53^4 2 \\ 6^2 43^5 \\ 65^3 43^2 \\ 65^2 43^3 2 \end{cases}$$

connected by the five relations

$$\begin{aligned} 6^2 5^2 3^3 &\equiv -\,654^2 3^4, \\ 6^2 5 3^4 2 &\equiv +\,654^2 3^4, \\ 6^2 4 3^5 &\equiv -\,654^2 3^4, \\ 6 5^3 4 3^2 &\equiv +\,2\,(654^2 3^4), \\ 6 5^2 4 3^3 2 &\equiv -\,2\,(654^2 3^4). \end{aligned}$$

So far therefore it has been proved that the generat perpetuants is $\frac{x^{31} + 0x^{32} + \dots}{2.3.4.5.6}$,

and for sextic syzygies (*vide* Prof. Cayley's paper)

$$\frac{x^6 + x^{13} - 2x^{16} - x^{18} + x^{31} + 0x^{32} + \dots}{2.3.4.5.6}.$$

There seems to be a very great probability that the true form of the numerator of the G. F. for perpetuants is monomial, *i. e.* simply x^{31}, but the way to show this does not seem clear; the question seems to be: 'Do any quintic forms exist of a weight superior to 25, which, not containing in their symbols the symbol $\widetilde{124}$, are not connected, by forms containing the symbol $\widetilde{124}$, with sextic syzygies?'

The possible forms are
$$\begin{aligned} &\widetilde{\varkappa 1 \mu}, \ (\varkappa > 1 \ \ \mu > 1), \\ &\widetilde{\varkappa\lambda 3}, \ (\varkappa > 1), \\ &\widetilde{\varkappa\lambda 2}, \ (\varkappa > 2), \end{aligned}$$

and it is easily seen from a consideration of the lettered syzygies '*ante*' that $\mu > 2$, the form $\widetilde{\varkappa 1 \mu}$ is impossible, but as regards the other forms there seem to be difficulties, and it is likely that the simple syzygies will have to be capitated, a work of great labor, in order to settle this point.

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